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Clustering of volatility as a multiscale phenomenon

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Abstract. The dynamics of prices in financial markets has been studied intensively both experimentally (data analysis) and theoretically (models). Nevertheless, a complete stochastic characterization of volatility is still lacking. What is well known is that absolute returns have memory on a long time range, this phenomenon is known as clustering of volatility. In this paper we show that volatility correlations are power-laws with a non-unique scaling exponent. This kind of multiscale phenomenology has some analogies with fully developed turbulence and disordered systems and it is now pointed out for financial series. Starting from historical returns series, we have also derived the volatility distribution, and the results are in agreement with a log-normal shape. In our study, we consider the New York Stock Exchange (NYSE), daily composite index closes (January 1966 to June 1998) and the US Dollar/Deutsche Mark (USD-DM) noon buying rates certified by the Federal Reserve Bank of New York (October 1989 to September 1998).

PACS. 02.50.-r Probability theory, stochastic processes, and statistics – 89.90.+n Other topics of general interest to physicists

1 Introduction

One of the most challenging problems in finance is the stochastic characterization of market returns. This topic not only has an academic relevance but also an obvious technical interest. Think, for example, of the option pricing models where distribution and correlations of volatility play a central role.

It is now well established that returns of the most important indices and foreign exchange markets have a distribution with fat tails. This means that the distribution of returns decays slower than a Gaussian, which implies that price processes are not simple random-walks. Moreover, returns distributions are uncorrelated for lags larger than a single day, in agreement with the hypothesis of efficient market. However, the distribution of volatility and its correlations are still poorly understood. What is known is that absolute returns (which are a measure of volatility) have memory on a long time range. This phenomenon is known in financial literature as clustering of volatility. Recent studies provide strong evidence for power-law correlations for absolute returns [1–6]. Notice that in the ARCH-GARCH approach, [7–9], volatility memory is longer than a single time step but decays exponentially implying that ARCH-GARCH modeling is inappropriate. Indeed, GARCH models have been extended in order to take into account this long memory property [2,10–12].

In this paper we analyze the daily returns of the New York Stock Exchange (NYSE) composite index from January 1966 to June 1998, and the US Dollar/Deutsche Mark (USD-DM) noon buying rates certified by the Federal Reserve Bank of New York from October 1989 to September 1998. We not only find that volatility correlations are power-laws on long time scales up to a year for NYSE index and six months for USD-DM exchange rate, but, more importantly, that they exhibit a non-unique exponent (multiscaling). This kind of multiscale phenomenology has some analogies with fully developed turbulence and disordered systems [13]. Indeed, we do not show the coexistence of many time-scales as in turbulence [14–16] but find that correlation memory has different lengths for different volatility sizes. This is consistent with a multifractal behaviour for stock markets [17].

The non-unique exponents for power-law correlations have been recently pointed out for financial series [18, 19]. Our result is based on the fluctuation analysis of a new class of variable that we call *generalized cumulative absolute returns*.

The second main result of the paper is the study of volatility probability distribution, which is derived by means of Fourier transform analysis. It is shown that it is well approximated by a log-normal distribution for NYSE index, while a log-normal shape is a reasonable fit only around the maximum for USD-DM rate.

The paper is organized as follows: in Section 2 we show that volatility has a long memory by considering the autocorrelation of absolute returns. Nevertheless,

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the power-law behaviour cannot be inferred by simply considering autocorrelations. In order to produce sharper evidence for the nature of the long memory phenomenon, in Section 3 we perform a scaling analysis on the standard deviation of a new class of observables, the generalized cumulative absolute returns. This analysis implies power-law correlations with a non-unique exponent. In Section 4 the attention is focused on volatility probability distribution, computed from returns data by means of Fourier transform analysis, which turns out to be log-normal at least for NYSE index. In Section 5 some final remarks can be found.

2 Correlations for returns

We consider the New York Stock Exchange (NYSE) daily composite index closes (January 1966 to June 1998) and the US Dollar/Deutsche Mark (USD-DM) noon buying rates certified by the Federal Reserve Bank of New York (October 1989 to September 1998). In the first case the dataset contains 8180 quotes, in the second 2264. The quantity we consider is the de-averaged daily return, defined as

$$r_t = \log \frac{S_{t+1}}{S_t} - \left\langle \log \frac{S_{t+1}}{S_t} \right\rangle \tag{1}$$

where S_t is the index quote or the exchange quote at time t. The time t ranges from 1 to N where N is the total number of quotes (8180 for the NYSE index and 2264 for the USD-DM exchange rate). The notation $\langle \cdot \rangle$ indicates the average over the whole sequence of N data.

As pointed out by several authors [20–23], the distribution of returns is leptokurtic. In [21], it was firstly proposed to be a symmetric Lévy stable distribution and more recently in [22] it is argued that the distribution is Lévy stable except for tails, which are approximately exponential. The estimation is that the shape of a Gaussian is recovered only on longer scales, typically for monthly returns [16,24–26].

Let us introduce the autocorrelation for returns, defined as

$$C(L) = \langle r_t r_{t+L} \rangle \tag{2}$$

since r_t is a zero mean process.

A direct numerical analysis of (2) for the NYSE index (Fig. 1a) and for USD-DM rate (Fig. 1b) shows that the returns autocorrelation is a vanishing quantity for all L. This simple evidence could induce us to the wrong conclusion that the description is complete, *i.e.* returns are i.i.d. variables whose distribution is a truncated Lévy. The situation is much more complicated: even if the returns autocorrelation vanishes, one cannot conclude that returns are independent variables. Independence implies that all functions of returns are uncorrelated variables. This is known to be false, in fact volatility has a long memory. On the other hand, the daily volatility is not directly observable, and information about it can be derived by means of absolute returns $|r_t|$.

It is useful to consider the following autocorrelation for powers of absolute returns

$$C(L, \gamma) = \langle |r_t|^\gamma \, |r_{t+L}|^\gamma \rangle - \langle |r_t|^\gamma \rangle \langle |r_{t+L}|^\gamma \rangle \cdot \tag{3}$$

This quantity is plotted for $\gamma = 1$ in Figure 1a (NYSE index) and in Figure 1b (USD-DM exchange rate). Unlike the returns autocorrelation, it turns out to be a non-vanishing quantity, at least up to $L \simeq 150$ (see [27,28] and the references therein). This is clear evidence that it is not correct to assume returns as independent random variables.

On the other hand, Figures 1 cannot give a satisfactory answer about the shape of absolute returns autocorrelations. In fact, data show a wide spread compatible with a different scaling hypothesis. In Figures 1 we have plotted two power-law functions with exponents derived by scaling analysis, which will be performed in the next section. The proposed interpolations are consistent with numerical data.

3 Scaling analysis

In the previous section, we have seen that consistent with the efficient market hypothesis, daily returns have no autocorrelations for lags larger than a single day. This fact can be also checked using scaling analysis. Consider the cumulative returns $\phi_t(L)$, defined as the sum of L successive returns $r_t, \ldots, r_{t+L-1}$ divided by L

$$\phi_t(L) = \frac{1}{L} \sum_{i=0}^{L-1} r_{t+i} = \frac{1}{L} \left[\log \frac{S_{t+L}}{S_t} - \left\langle \log \frac{S_{t+L}}{S_t} \right\rangle \right] \cdot \tag{4}$$

One can define N/L non overlapping variables of this type, and compute the associated variance $\mathrm{Var}\,(\phi(L))$. Assuming that r_t are uncorrelated (or short range correlated), it follows that $\mathrm{Var}\,(\phi(L))$ has a power-law behaviour proportional to $L^{-\alpha}$ with exponent $\alpha = 1$ for large L (see Appendix A), *i.e.*

$$\mathrm{Var}\,(\phi(L)) \sim L^{-1}. \tag{5}$$

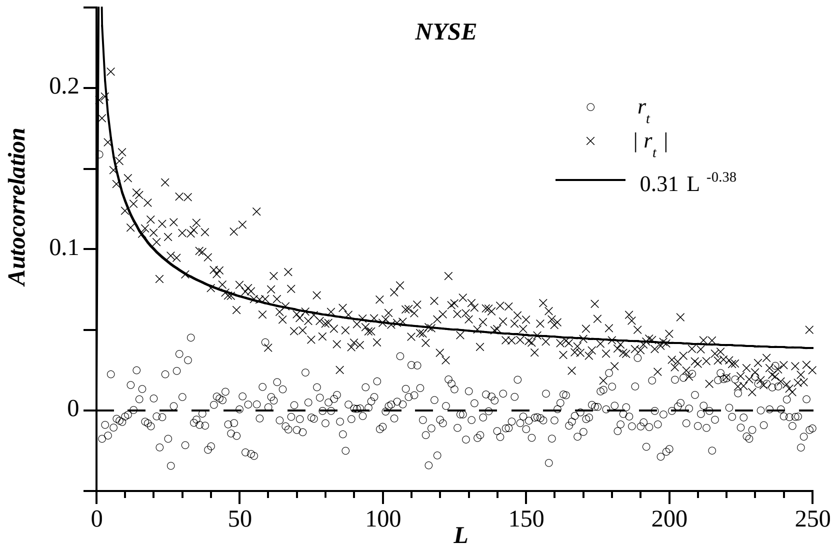
The exponent α both for the NYSE index and USD-DM exchange market turns out to be around 1 (see Figs. 2 and also see [16,23]), confirming that returns are uncorrelated.

Conversely, this is not true for other quantities related to absolute returns. In order to perform the appropriate scaling analysis, let us introduce the generalized cumulative absolute returns defined as the sum of L successive returns $|r_t|^\gamma, \ldots, |r_{t+L-1}|^\gamma$, divided by L

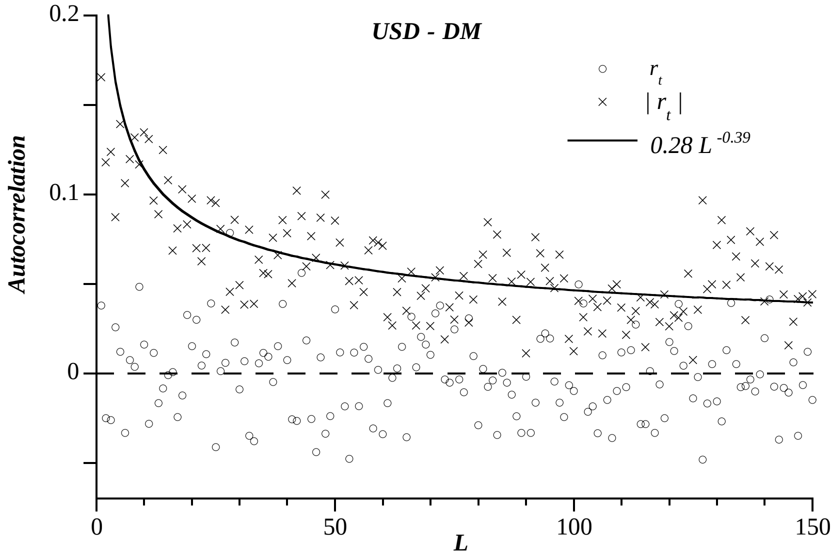
$$\phi_t(L, \gamma) = \frac{1}{L} \sum_{i=0}^{L-1} |r_{t+i}|^\gamma \tag{6}$$

where γ is a real exponent and again these quantities are not overlapping.

In Appendix A we show that if the autocorrelation for powers of absolute returns (3) exhibits a power-law with



(a)



(b)

Fig. 1. Autocorrelation $C(L,1)$ of $|r_t|$ (crosses) as a function of the correlation length L, compared with the returns autocorrelation $C(L)$ (circles), for: (a) NYSE index; (b) USD-DM exchange rate. The scale, if fixed by autocorrelation, equals 1 at $L=0$. The data for absolute returns are in agreement with a power-law with exponent, respectively, $\alpha(1) \simeq 0.38$ for NYSE index and $\alpha(1) \simeq 0.39$ for USD-DM rate, which are derived by independent scaling analysis.

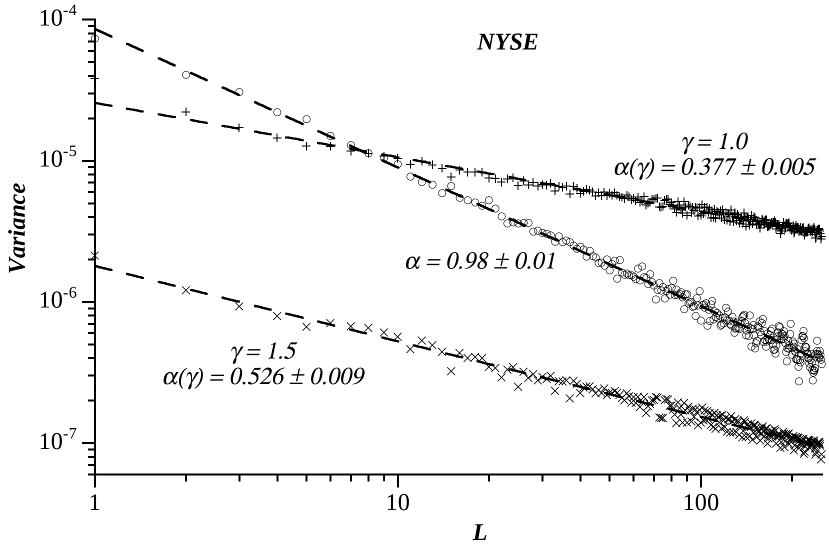
exponent $\alpha(\gamma) \leq 1$ for large L, *i.e.* $C(L,\gamma) \sim L^{-\alpha(\gamma)}$, it would be implied that

$$\mathrm{Var}\,(\phi(L,\gamma)) \sim L^{-\alpha(\gamma)}. \quad (7)$$

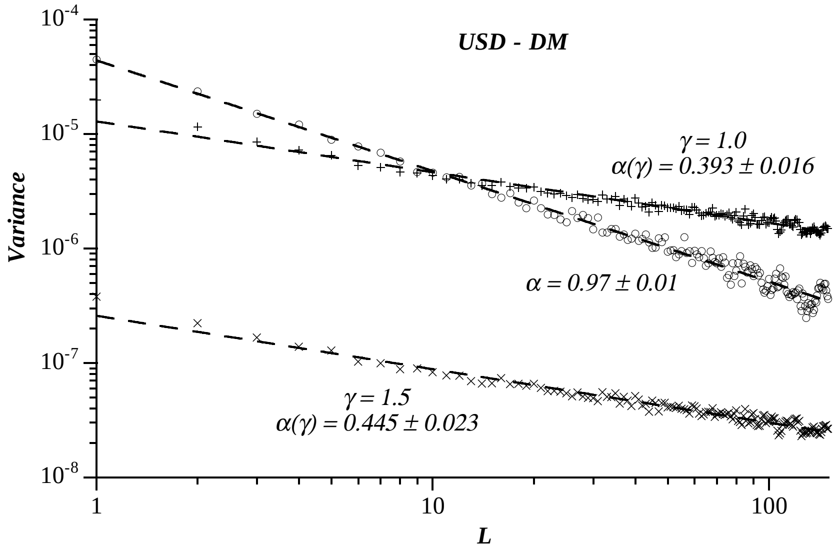
However, if the $|r_t|^\gamma$ are short-range correlated or power-law correlated with an exponent $\alpha(\gamma) > 1$, we would not detect anomalous scaling in the analysis of variance, *i.e.* $\mathrm{Var}\,(\phi(L,\gamma)) \sim L^{-1}$.

Our numerical analysis shows very sharply an anomalous power-law behaviour with exponent $\alpha < 1$, after a very short transient time, in the range up to one year ($L = 250$) for NYSE index (Fig. 2a), and up to six months ($L = 150$) for the USD-DM exchange market (Fig. 2b). For larger L, the number of non overlapping variables $\phi(L,\gamma)$ becomes too small for a statistical analysis, as revealed also by the increasing fluctuations on variance $\mathrm{Var}\,(\phi(L,\gamma))$ as a function of L. The best fit straight lines are performed in the ranges, $10 \leq L \leq 250$ for the NYSE index, and $10 \leq L \leq 150$ for the USD-DM rate.

The crucial result is that $\alpha(\gamma)$ is not a constant function of γ, showing the presence of different anomalous scales. The interpretation is that different values of γ select different typical fluctuation sizes, any of them being



(a)



(b)

Fig. 2. Variance $\mathrm{Var}\,(\phi(L,\gamma)) \sim L^{-\alpha(\gamma)}$ of the generalized cumulative absolute returns as a function of L on log-log scales for $\gamma = 1$ (crosses) and $\gamma = 1.5$ (slanting crosses), compared with the variance $\mathrm{Var}\,(\phi(L)) \sim L^{-\alpha}$ of the cumulative returns (circles), for: (a) NYSE index; (b) USD-DM exchange rate. The exponents of the best fit straight lines (dashed lines) are: $\alpha(1) = 0.377 \pm 0.005$, $\alpha(1.5) = 0.526 \pm 0.009$ and $\alpha = 0.98 \pm 0.01$ for the NYSE index; $\alpha(1) = 0.393 \pm 0.016$, $\alpha(1.5) = 0.445 \pm 0.023$ and $\alpha = 0.97 \pm 0.01$ for the USD-DM exchange rate.

power-law correlated with a different exponent. The case $\gamma = 0$ corresponds to a cumulative logarithm of absolute returns. Approximately, in the region $\gamma \geq 4$, the averages are dominated by only a few events, corresponding to very large returns, therefore the statistics become insufficient.

In Figure 3, $\alpha(\gamma)$ is plotted as a function of γ with error bars for both cases. In the NYSE index case, the exponent $\alpha(\gamma)$ exhibits a large spread, reaching an ordinary scaling exponent $\alpha(\gamma) = 1$ for $\gamma \simeq 4$. Contrarily, the USD-DM exponent turns out to be less variable, but its anomalous scaling persists at least up to $\gamma = 4$.

We would like to stress that the scaling analysis in Figures 2 definitively proves the power-law behaviour and precisely determines the coefficients $\alpha(\gamma)$, while a direct analysis of the autocorrelations (as in Figs. 1) would not have provided an analogous clear evidence for multiscale power-law behaviour, since the data show a wide spread compatible with different scaling hypotheses.

The anomalous power-law scaling can be eventually tested against the plot of autocorrelations. For instance, the autocorrelations of r_t and of $|r_t|$ are plotted in Figures 1 as a function of the correlation length L, and

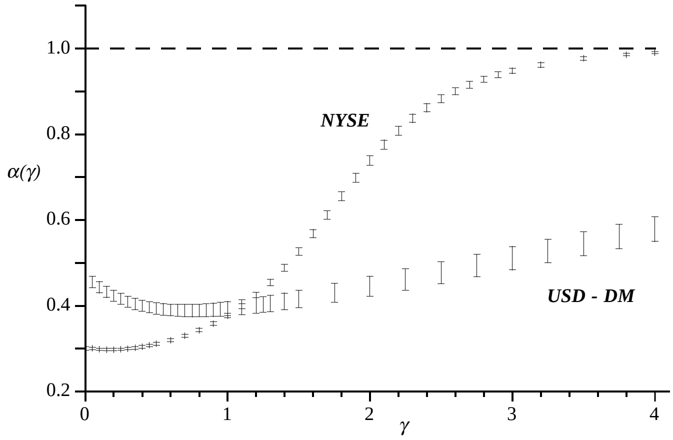


Fig. 3. Scaling exponent $\alpha(\gamma)$ of the variance $\mathrm{Var}\,(\phi(L,\gamma)) \sim L^{-\alpha(\gamma)}$ as a function of γ for NYSE index and USD-DM rate, where the bars represent the errors over the best fits. An anomalous scaling ($\alpha(\gamma) < 1$) is shown for both cases.

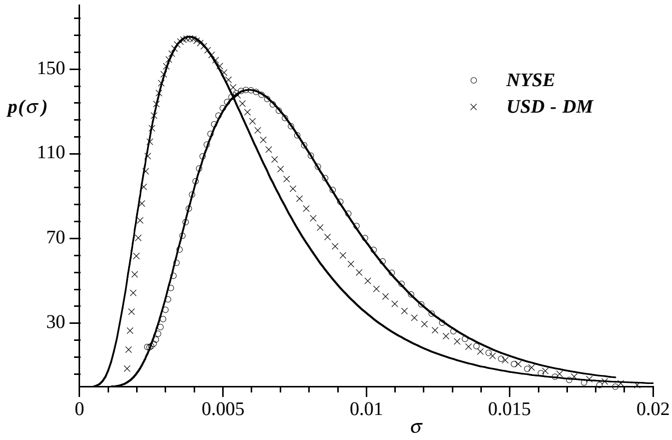


Fig. 4. Probability distribution $p(\sigma)$ of volatility for NYSE index (circles) and USD-DM exchange rate (crosses) fitted by log-normal distributions (9) with, $m = -4.94 \pm 0.01$ and $s = 0.44 \pm 0.01$ for the NYSE index (fit performed in the range $0.0035 \le \sigma \le 0.01$), and $m = -5.27 \pm 0.01$ and $s = 0.54 \pm 0.01$ for the USD-DM rate (fit range $0.0025 \le \sigma \le 0.005$).

the full line, which is in a good agreement with the data, is not a best fit but a power-law whose exponent $\alpha(1)$ is obtained by the scaling analysis of the variance.

4 Distribution of volatility

All the discussion in the previous section concerns absolute returns. An obvious question is: "what is the relation with volatility?". The answer is not completely trivial, since from an operative point of view, the volatility is often assumed to coincide with the intra-day absolute cumulative return, or alternatively with the implied volatility which can be extracted from option prices.

Our point of view is that the exact definition of volatility cannot be independent from the theoretical framework. It is usually assumed that the volatility σ_t is defined by

$$r_t = \sigma_t \, \omega_t \qquad (8)$$

where the ω_t are identical independently distributed random variables with vanishing average and unitary variance. The usual choice for the distribution of the ω_t is the normal Gaussian. This picture is completed by assuming the probabilistic independence between σ_t and ω_t.

In other terms, the returns series can be considered as a realization of a random process based on a zero mean Gaussian, with a standard deviation σ_t that changes at each time step. According to the above definition, all the scaling properties we have found on absolute returns directly apply to volatility.

Volatility σ_t is a *hidden* variable, since we can directly evaluate only daily returns. Nevertheless, in Appendix B we show how to derive the volatility probability distribution $p(\sigma)$ starting from the returns series. The key point is to move the problem into the space of the characteristic functions (Fourier transforms).

The probability distribution $p(\sigma)$ is plotted in Figure 4, both for the NYSE index and the USD-DM exchange rate. The results corresponding to extreme values of volatility ($\sigma \simeq 0$ and $\sigma \simeq 0.02$) are not confident due to insufficient statistics.

The astonishing fact is that the NYSE volatility distribution is well fitted by a log-normal distribution [28,29]

$$p(\sigma) = \frac{1}{\sqrt{2\pi}\; s\; \sigma} \quad \exp -\frac{1}{2}\left(\frac{\log\sigma - m}{s}\right)^2 \cdot \qquad (9)$$

The fit is performed in the range $0.0035 \le \sigma \le 0.01$ and gives $m = -4.94 \pm 0.01$ and $s = 0.44 \pm 0.01$, while the USD-DM volatility distribution is consistent with a log-normal distribution with $m = -5.27 \pm 0.01$ and $s = 0.54 \pm 0.01$ only in a narrow region around the maximum ($0.0025 \le \sigma \le 0.005$).

This unexpected log-normal shape for the volatility distribution suggests the existence of some underlying multiplicative process for volatility, at least for the NYSE index. This result implies that not only are the index prices multiplicative processes, but also the associated returns. On the other hand, the USD-DM rate analysis might be affected by insufficient statistics problems, leading to an over-estimation of the distribution tail in the range $\sigma \simeq 0.01$. Under this hypothesis, a log-normal shape could be consistent with the USD-DM volatility distribution, and an underlying multiplicative process might be present also for foreign exchange returns.

Reasonable tentatives to explain this peculiar behaviour for the volatility distribution can be found in [15] where a multiplicative cascade process for volatility is proposed borrowing well-known arguments from turbulence theory. In [30–32] it is shown how a random multiplicative process model gives rise to power-laws and exponential tails. Power-laws also may indicate a self-organized critical phenomenology as proposed in [33,34].

5 Conclusions

The first result we have found is that the scaling of variance of the generalized cumulative absolute returns is a power-law with a non-unique exponent for both the NYSE daily index and the USD-DM exchange rate. This fact implies power-law correlations whose exponent depends on the variable being considered. The main theoretical consequence is that models with exponential correlations, like ARCH-GARCH, fail in describing the dynamics of financial markets, and that new models should account for the coexistence of long memory with different scales.

The second result is that volatility distribution is log-normal, at least for NYSE index. This fact suggests that volatility itself evolves as a multiplicative process.

These two results show the existence of an underlying process that drives daily returns and indicates that new modelizations of financial markets have to look to returns as a subordinate process of volatility.

Appendix A

In this appendix we show that if the correlations $C(L,\gamma)$ exhibit a long range memory, $C(L,\gamma) \sim L^{-\alpha(\gamma)}$ with $\alpha < 1$, then also the variance $\mathrm{Var}(\phi(L,\gamma))$ of the *generalized cumulative absolute returns* behaves at large L as $L^{-\alpha(\gamma)}$.

The explicit expression of variance is

$$\mathrm{Var}\,(\phi(L,\gamma)) = \frac{1}{L^2}\sum_{i=1}^{L}\sum_{j=1}^{L}\langle|r_{t+i}|^\gamma |r_{t+j}|^\gamma\rangle - \langle|r_{t+i}|^\gamma\rangle\langle|r_{t+j}|^\gamma\rangle\;.$$

Taking into account that r_t is a stationary process, and using the definition of $C(L,\gamma)$ (3), one has:

$$\mathrm{Var}\,(\phi(L,\gamma)) = \frac{1}{L}C(0,\gamma) + \frac{2}{L^2}\sum_{L\geq i>j\geq 1} C(i-j,\gamma)$$

where

$$C(0,\gamma) = \langle|r_t|^{2\gamma}\rangle - \langle|r_t|^\gamma\rangle^2.$$

The previous expression can be rewritten as

$$\mathrm{Var}\,(\phi(L,\gamma)) = \frac{1}{L}C(0,\gamma) + \frac{2}{L^2}\sum_{i=1}^{L-1}(L-i)\;C(i,\gamma).$$

Under the hypothesis $C(L,\gamma) \sim L^{-\alpha(\gamma)}$, one has for large L

$$\frac{2}{L^2}\sum_{i=1}^{L-1}(L-i)\;C(i,\gamma) \sim L^{-\alpha(\gamma)}$$

which leads to

$$\mathrm{Var}\,(\phi(L,\gamma)) = O(L^{-1}) + O(L^{-\alpha(\gamma)}).$$

For our data $\alpha(\gamma) \leq 1$, and then

$$\mathrm{Var}\,(\phi(L,\gamma)) \sim L^{-\alpha(\gamma)}.$$

Contrarily, if $\alpha(\gamma) > 1$ or greater, correlations exhibit a faster decay, the variance $\mathrm{Var}(\phi(L,\gamma))$ would be a power-law with a scaling exponent equal to 1.

A similar sketch can be repeated for the cumulative returns $\phi(L)$. In this case since the correlation has a fast decay, we have

$$\mathrm{Var}\,(\phi(L,\gamma)) \sim L^{-1}.$$

Appendix B

Let us introduce the variables $\mathcal{R}_t, \mathcal{S}_t, \mathcal{W}_t$, defined as

$$\begin{aligned}\mathcal{R}_t &= \log|r_t|\\ \mathcal{S}_t &= \log\sigma_t\\ \mathcal{W}_t &= \log|\omega_t|\end{aligned}$$

which are related among them by virtue of (8) by

$$\mathcal{R}_t = \mathcal{S}_t + \mathcal{W}_t.$$

For the associated probability distributions (respectively $Q(\mathcal{R}), P(\mathcal{S}), T(\mathcal{W})$) the following relation holds

$$Q(\mathcal{R}) = \int_{-\infty}^{+\infty} \mathrm{d}\mathcal{S}\; P(\mathcal{S})T(\mathcal{R}-\mathcal{S}). \tag{10}$$

The distribution $P(\mathcal{S})$ retains full information on the volatility probability distribution $p(\sigma)$, since $p(\sigma) = P(\log\sigma)/\sigma$.

In order to derive from (10) an explicit expression for $P(\mathcal{S})$, it is convenient to consider the characteristic functions (Fourier transforms) $\tilde{Q}(\tilde{\mathcal{R}}), \tilde{P}(\tilde{\mathcal{S}}), \tilde{T}(\tilde{\mathcal{W}})$ of $Q(\mathcal{R}), P(\mathcal{S}), T(\mathcal{W})$. In fact, the following simple relation holds

$$\tilde{Q}(\tilde{\mathcal{S}}) = \tilde{P}(\tilde{\mathcal{S}})\tilde{T}(\tilde{\mathcal{S}})$$

and the inverse Fourier transform gives

$$P(\mathcal{S}) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}\tilde{\mathcal{S}}\;\frac{\tilde{Q}(\tilde{\mathcal{S}})}{\tilde{T}(\tilde{\mathcal{S}})}\;\mathrm{e}^{\mathrm{i}\mathcal{S}\tilde{\mathcal{S}}}.$$

Notice that $\tilde{Q}(\tilde{\mathcal{S}})$ and $\tilde{T}(\tilde{\mathcal{S}})$ are complex objects, but we may consider only the real part of the integrand, since the result of the integration has to be real

$$P(\mathcal{S}) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}\tilde{\mathcal{S}}\;\mathrm{Re}\left[\frac{\tilde{Q}(\tilde{\mathcal{S}})}{\tilde{T}(\tilde{\mathcal{S}})}\;\mathrm{e}^{\mathrm{i}\mathcal{S}\tilde{\mathcal{S}}}\right] \tag{11}$$

where

$$\mathrm{Re}\left[\frac{\tilde{Q}(\tilde{\mathcal{S}})}{\tilde{T}(\tilde{\mathcal{S}})}\mathrm{e}^{\mathrm{i}\mathcal{S}\tilde{\mathcal{S}}}\right] = \frac{\left(\mathrm{Re}\,\tilde{Q}\;\mathrm{Re}\,\tilde{T} + \mathrm{Im}\,\tilde{Q}\;\mathrm{Im}\,\tilde{T}\right)\cos(\mathcal{S}\tilde{\mathcal{S}})}{(\mathrm{Re}\,\tilde{T})^2 + (\mathrm{Im}\,\tilde{T})^2} + \frac{\left(\mathrm{Re}\,\tilde{Q}\;\mathrm{Im}\,\tilde{T} - \mathrm{Im}\,\tilde{Q}\;\mathrm{Re}\,\tilde{T}\right)\sin(\mathcal{S}\tilde{\mathcal{S}})}{(\mathrm{Re}\,\tilde{T})^2 + (\mathrm{Im}\,\tilde{T})^2}\,.$$

From a practical point of view, $\mathrm{Re}\,\tilde{Q}(\tilde{\mathcal{S}})$ and $\mathrm{Im}\,\tilde{Q}(\tilde{\mathcal{S}})$ can be directly computed from the returns series

$$\mathrm{Re}\,\tilde{Q}(\tilde{\mathcal{S}}) = \int_{-\infty}^{+\infty} \mathrm{d}\mathcal{R}\; Q(\mathcal{R})\;\cos(\tilde{\mathcal{S}}\mathcal{R}) \;\simeq\; \frac{1}{N}\sum_{t=1}^{N}\cos(\tilde{\mathcal{S}}\mathcal{R}_t)$$

$$\mathrm{Im}\,\tilde{Q}(\tilde{\mathcal{S}}) = \int_{-\infty}^{+\infty} \mathrm{d}\mathcal{R}\; Q(\mathcal{R})\;\sin(\tilde{\mathcal{S}}\mathcal{R}) \;\simeq\; \frac{1}{N}\sum_{t=1}^{N}\sin(\tilde{\mathcal{S}}\mathcal{R}_t).$$

The Fourier transforms $\mathrm{Re}\,\tilde{T}(\tilde{\mathcal{S}})$ and $\mathrm{Im}\,\tilde{T}(\tilde{\mathcal{S}})$ can be evaluated numerically starting from their definitions:

$$\mathrm{Re}\,\tilde{T}(\tilde{\mathcal{S}}) = \int_{-\infty}^{+\infty} \mathrm{d}\mathcal{R}\; T(\mathcal{R})\;\cos(\tilde{\mathcal{S}}\mathcal{R})$$

$$\mathrm{Im}\,\tilde{T}(\tilde{\mathcal{S}}) = \int_{-\infty}^{+\infty} \mathrm{d}\mathcal{R}\; T(\mathcal{R})\;\sin(\tilde{\mathcal{S}}\mathcal{R})$$

where

$$T(\mathcal{R}) = \sqrt{\frac{2}{\pi}}\;\mathrm{e}^{\mathcal{R}-\frac{1}{2}\mathrm{e}^{2\mathcal{R}}}.$$

Finally, the probability distribution $P(\mathcal{S})$, and then $p(\sigma)$, can be computed *via* the numerical evaluation of integral (11).

The key step of this procedure is the numerical inverse Fourier transform, therefore the delicate point is the evaluation of the tails of the probability distribution $P(\mathcal{S})$, where the limited number of data leads to spurious fluctuations.

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